From scaling to multiscaling in the stochastic Burgers equation

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We investigate the scaling behavior of the structure functions, $S_q(r)=\langle[u(r)-u(0)]^q\rangle\propto|r|^{\zeta q}$, in the stochastic Burgers equation as a function of the exponent β that characterizes the scale of noise correlations, for $0<\beta<-1$. We analyze the exact equations satisfied by $S_q(r)$ $(q=3,4,5)$ based on certain *ansätze*. For small negative β Kolmogorov-like scaling with $\zeta_q=-q\beta/3$ is obtained; as $\beta\rightarrow-1$ an increasing multifractal structure occurs with bifractality for $\beta<-1$. We determine ζ_4 and ζ_5, which are piecewise continuous and the associated multifractal scaling exponents. [S1063-651X(97)13010-0]

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One aspect of fully developed turbulence that has been studied extensively, theoretically, and experimentally is the occurrence of multifractality [1]; the focus has been on the velocity structure functions, $S_q(r)\equiv\langle[u(x+r/2,t)-u(x-r/2,t)]^q\rangle$. For r in the inertial range and for large Reynolds numbers (as $\nu\rightarrow0$), the behavior of the structure functions is given by

$$S_q(r)\sim|r|^{\zeta_q}, \tag{1}$$

where we have restricted our attention to one dimension for simplicity. The deviation from linearity of ζ_q, i.e., $\zeta_q\neq cq$, signals multifractal behavior.

It is useful to have simple models in which one can study various theoretical issues associated with scaling and multifractality, such as the influence of infrared (system size) and ultraviolet (viscous dissipation scale) cutoffs on scaling behavior. The stochastic Burgers equation in one dimension, a recent focus of attention [2–6] provides an interesting arena to study some of these theoretical issues; it is given by

$$\partial u/\partial t+u\,\partial u/\partial x=\nu\nabla^2u+\eta(x,t), \tag{2}$$

where $u(x,t)$ is the velocity field, ν the viscosity, and $\eta(x,t)$ is a Gaussian noise, with zero mean, and correlations in k space determined by

$$\langle\hat{\eta}(k,t)\,\hat{\eta}(k',t')\rangle=2\hat{D}(k)\,\delta_{k,-k'}\,\delta(t-t'). \tag{3}$$

In this paper we will study the case in which the noise variance $\hat{D}(k)$ exhibits power-law behavior, $\hat{D}(k)=D_0|k|^\beta$ with $\beta<0$; this model has been investigated both for $\beta=-1$ [3], and for $2\geqslant\beta\geqslant-1$ [4]. When β is positive, the model has been studied in the interface representation by Medina *et al.* [7]. Here shocks are not important and the structure factors approach constant values at large distances consistent with Gaussian behavior. Simple scaling behavior is obtained. However, when β is negative the profiles of the velocity field in the statistically stationary state show well-defined shocks in contrast to $\beta>0$. We have found numerically that for $0>\beta\geqslant-1$ the exponents ζ_q depend on β and are not linear in q [4].

We summarize the main results of this paper. Our results are based on the exact equations satisfied by $S_q(r)$ for $q=3,4,5$ that we analyze based on assumptions that we will describe. We use numerical simulations based on a pseudospectral code (see Ref. [4] and [8]) to provide plausible evidence for some of our analytic work. We demonstrate that for $\beta<-1$ the behavior is actually that of a system with cutoff noise, i.e., $\zeta_q=1$ for $q>2$, analogous to an extreme form of multifractality dubbed bifractality [1,9]. The following scenario emerges for $-1<\beta<0$ that connects simple scaling behavior on the one hand and bifractality on the other. For small negative β $S_q(r)$ exhibits (Kolmogorov-like) scaling with $\zeta_q=-q\beta/3$ until at some β it deviates from this; the deviation occurs closer to $\beta=0$, the larger the value of q. Conversely, at any β the naive scaling exponent is exhibited by S_q for small values of q; the deviation occurs for larger and larger values of q the closer β is to 0. Eventually, for $\beta\leqslant-3/2$, the system exhibits bifractality. In addition, we show that the inner cutoff scale of the inertial range depends on the order of the structure function considered and deduce possible forms of the associated multifractal scaling exponents [10].

This rich multiscaling behavior is partly due to the occurrence of shocks and partly due to the flux of energy, $\Pi(K)$, not being independent of the scale, i.e., energy is supplied at all scales. Recall that $\Pi(k)$ is defined as the contribution of the nonlinear term to $-(1/2L^2)\partial_t\Sigma_{|k|<K}\langle\hat{u}(k)\hat{u}(-k)\rangle$. Using the fact that in the inertial range viscous dissipation is negligible and the flux is provided by the stochastic noise we find that the flux is given by $\Pi(K)\propto\Sigma_{1/L}^{K}\hat{D}(k)$. From the form of the stochastic noise it is easy to see that for $0>\beta>-1$, $\Pi(K)\propto(D_0/L)K^{1+\beta}$ where L is the system size. In this respect this system differs from fully developed, three-dimensional turbulence, where $\Pi(K)$ is constant in the inertial range. However, for $\beta<-1$ the energy flux is a constant proportional to $L^{-1-\beta}$ since the above expression for the flux is dominated by the infrared cutoff; energy is being pumped into the system at large scales and dissipated in shocks. At $\beta=-1$, the flux is a constant to logarithmic accuracy.

We begin with a discussion of the behavior of $S_2(r)$, which is directly related to $E(k)=\langle\hat{u}(k)\hat{u}(-k)\rangle$. If $E(k)\propto|k|^{-\sigma}$ then $\zeta_2=\sigma-1$. The numerical evidence leads to the following picture for $E(k)$ for $-1<\beta<0$: At small length scales we obtain free-field behavior with $\sigma=\beta-2$, which crosses over to inertial-range behavior, at a length scale de-

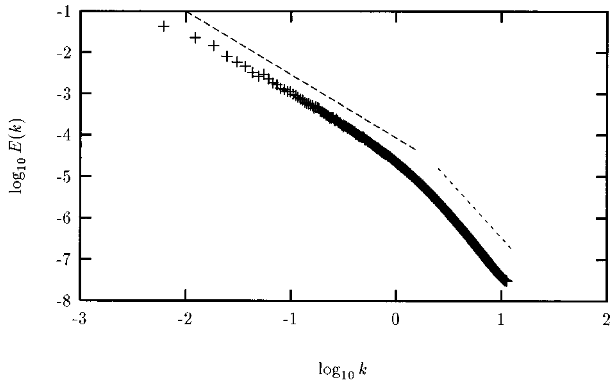

FIG. 1. $\log_{10} E(k)$ vs $\log_{10} k$, where $E(k)=\langle\hat{u}(k)\hat{u}(-k)\rangle$ is the energy spectrum for $\beta=-0.80$, $\nu=0.06$, and $D=1.0\times10^{-6}$. The upper straight line, drawn for comparison, has a slope given by $-1+2\beta/3$, which is equal to -1.53 for the value of β considered. The lower line corresponds to free-field behavior with an exponent of $-2+\beta$. The wave vector k is in units of the basic interval in k space, namely, $2\pi/L$, where the system size is given by $L=1024$.

noted by l_c, namely, to $\sigma=1-2\beta/3$; the latter exponent is the one obtained by naive extrapolation of the fixed-point behavior obtained for positive β in Ref. [7].

We can give a heuristic explanation of this behavior based on two ingredients: (i) Free-field behavior occurs up to a length scale l_c at which the appropriate nonlinear coupling becomes of order unity under the renormalization group transformation; to leading order around the free-field fixed point one finds that l_c is given by $\nu\propto l_c^{1-\beta/3}$. (ii) In the inertial range, the input energy ''cascades'' to smaller scales and we can define a dissipation wave number K_c at which the dissipation, defined by $2\nu\int_0^{K_c}dk\ k^2\langle\hat{u}(k)\hat{u}(-k)\rangle$, becomes comparable to the energy flux. Substituting the scaling form $\langle\hat{u}(k)\hat{u}(-k)\rangle\propto|k|^{-\sigma}$ yields the result that the dissipation (up to K_c) is proportional to $\nu K_c^{3-\sigma}$; since this is comparable to the energy flux that is given by $K_c^{1+\beta}$, we conclude that $\nu\propto K_c^{\sigma-2+\beta}$. The reasonable identification of this edge of the inertial range, K_c, with the inverse of $l_c\approx O(\delta)$, at which free-field behavior begins, leads to $\sigma=1-2\beta/3$.

We thus have [5] for negative β,

$$S_2(r)\sim|r|^{-2\beta/3}. \tag{4}$$

Note that this behavior is obtained for $l_c<r<L_c$ where the outer scale is denoted by L_c and the inner scale l_c is of the order of the shock thickness δ. We note that this behavior is most easily seen numerically by studying $E(k)\propto|k|^{-\sigma}$; in Fig. 1 we show our results for $\beta=-0.80$ that are consistent with $\sigma=1-2\beta/3$ at long wavelengths and free-field behavior at short distances. We have also found that the long-wavelength behavior of S_2 for which theoretical justification was offered above persists down to $\beta=-1.50$.

We discuss next $S_3(r)$, which obeys the analog of the von Karman–Howarth relation given by

$$\frac{1}{6}\frac{dS_3(r)}{dr}=\nu\frac{d^2S_2(r)}{dr^2}-\frac{1}{L^2}\sum_k \hat{D}(k)\cos kr. \tag{5}$$

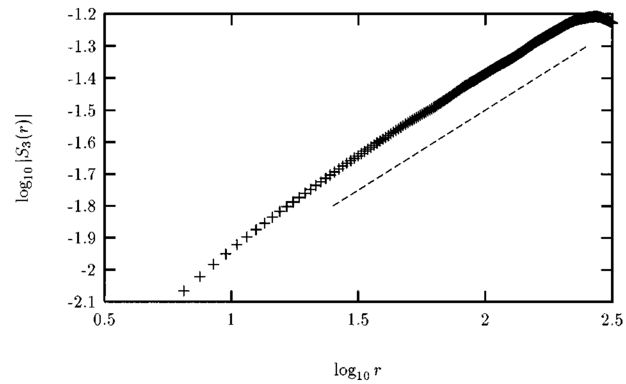

FIG. 2. $\log_{10}|S_3(r)|$ vs $\log_{10} r$. The straight line, drawn for comparison, has a slope equal to the predicted value -0.5. Here $\beta=-0.5$, $\nu=0.075$, and $D=1.0\times10^{-7}$.

In the limit $\nu\to0$ the first term is negligible compared to the term involving the noise $\eta(x,t)$. A straightforward analysis yields for $0>\beta>-1$

$$S_3(r)\propto\mathrm{sgn}(r)|r|^{-\beta}. \tag{6}$$

In Fig. 2 we display numerical results for $\beta=-0.5$ that are nicely consistent with the theoretical results given above. For $\beta<-1$, imposing an explicit cutoff on the noise spectrum at L_c, we find

$$S_3(r)=C_1\frac{r}{L_c}+C_2\left(\frac{r}{L_c}\right)^{-\beta}, \tag{7}$$

which for $r<L_c$ is dominated by the linear term. This means that the results are similar to those of the cutoff theory. Consequently, one can argue following Ref. [5] that higher-order structure functions also behave linearly. We have verified this numerically [11]. For $\beta>-1$ the subdominant term in Eq. (7) becomes the dominant one. Thus the behavior of S_3 interpolates between constant behavior at $\beta=0$, and linear behavior for $\beta<-1$.

The rest of our analysis is based on the equations for $S_4(r)$ and $S_5(r)$. The general result was presented in Ref. [5]. For a statistically stationary, homogeneous state the equations, using the notation that will be employed in the rest of the paper, $x_1=x+r/2$, $x_2=x-r/2$, $u_1=u(x_1)$, $u_2=u(x_2)$, and $\partial_i=\partial/\partial x_i$ for $i=1,2$ are

$$\frac{1}{6}\frac{dS_4(r)}{dr}=\frac{2\nu}{3}\frac{d^2S_3}{dr^2}-2\langle[\epsilon_1+\epsilon_2][u_1-u_2]\rangle \tag{8}$$

and

$$\frac{1}{40}\frac{dS_5(r)}{dr}=-\frac{1}{2L^2}\sum_k\hat{D}(k)\cos(kr)\langle(u_1-u_2)^2\rangle$$
$$-\frac{1}{3}[\langle[\epsilon_1+\epsilon_2](u_1-u_2)^2\rangle$$
$$-\langle[\epsilon_1+\epsilon_2]\rangle\langle(u_1-u_2)^2\rangle], \tag{9}$$

where $\epsilon_1=\nu(\partial_1u_1)^2$ and $\epsilon_2=\nu(\partial_2u_2)^2$. These and analogous results for S_q for $q>5$ show that their behavior is determined by the form of the operator product expansion

(OPE) for (the subtracted part of) $\langle(\epsilon_1+\epsilon_2)(u_1-u_2)^{q-3}\rangle$. If we assume that $\langle\epsilon_1(u_1-u_2)^m\rangle-\langle\epsilon_1\rangle\langle(u_1-u_2)^m\rangle$ behaves as $|r|^{-\mu_m}$ then the equations allow the identification [12] $\mu_m=\zeta_{m+3}-1$.

We will now argue, by examining Eqs. (8) and (9), that for negative β close to zero and for q not too large, $S_q(r)$ displays power-law behavior in the inertial range with

$$S_q(r)\sim|r|^{-q\beta/3}. \tag{10}$$

This behavior is subdominant for $\beta>0$, where all S_q behave as constants at large enough distance. For S_3, this relation is valid for all $-1<\beta<0$ as shown above. For $q>3$ Eq. (10) cannot be true for all $\beta<0$, since $\zeta_n\rightarrow 1$ as $\beta\rightarrow -1$. In higher-order structure factors one source of the scaling term is the term $S_{q-3}(r)\Sigma_k\hat{D}(k)\cos(kr)$ in the equation for $dS_q(r)/dr$. Since $\Sigma\hat{D}(k)\cos(kr)\propto|r|^{-\beta-1}$, scaling behavior for S_{q-3} yields scaling for S_q [13]. More generally a naive scaling argument with the scaling dimension of $h=-\beta/3$ for u and $-1+3h$ for ϵ leads to the result in Eq. (10). Note that there is a single multifractal exponent $h=-\beta/3$.

We discuss below how these exponents change to multifractal exponents as β is varied. In the absence of a complete understanding of the hierarchy of operators in the theory that determines the operator product expansion for all β we make certain assumptions to determine explicit expressions for ζ_4 and ζ_5 consistent with our expectations in the limits when $\beta=0$ and $\beta=-1$. These correspond to specific choices for the fusion rules for the (singular) operators that enter the Eqs. (8) and (9).

We deduce the behavior of $\langle(\epsilon_1+\epsilon_2)(u_1-u_2)\rangle$ that enters Eq. (8) as follows: note that for fixed ν it is proportional to

$$\int dk\int dq q(k+q)\sin(kr)\langle\hat{u}(k)\hat{u}(q)\hat{u}(-k-q)\rangle.$$

For the three-point function in k space, $\hat{s}_3(k_1,k_2,k_3)=\langle\hat{u}(k_1)\hat{u}(k_2)\hat{u}(k_3)\rangle$, we conjecture a form that is consistent with the known behavior of S_3 and use this to perform the k integrals with an explicit ultraviolet cutoff denoted by δ_4; we will assume that this dominates the scaling behavior of S_4. We conjecture the following symmetric form for the three-point function

$$\hat{s}_3(k_1,k_2,k_3)\sim|k_1|^{\mu_1}|k_2|^{\mu_2}|k_3|^{\mu_3}+(\text{permutations}). \tag{11}$$

With $\mu_1+\mu_2+\mu_3=\beta-2$ one recovers the behavior of S_3, namely, $S_3\sim|r|^{-\beta}$ for $-1<\beta<0$. Using this expression for the three-point function in Eq. (8) for S_4, a consistent solution is obtained when $\mu_1=\mu_2=\mu_3=(\beta-2)/3$; we find that this term contributes $S_4\sim\nu|r|^{1/3+(1-\beta)/3}/\delta_4^{1+2(1+\beta)/3}$. In the limit $\nu\rightarrow 0$, one must have $\delta_4\rightarrow 0$ such that

$$\nu\sim\delta_4^{1+2(1+\beta)/3}. \tag{12}$$

This analysis yields $\zeta_4=(2-\beta)/3$, where ζ_4 is the exponent of r in the above expression of S_4; since we seek scaling behavior for $r<L_c$ where L_c is the outer cutoff this behavior dominates over scaling with the exponent $-4\beta/3$ for $-1<\beta<-2/3$. Note that ζ_4 extrapolates to 1 at $\beta=-1$ as expected. The variation of ζ_4 with β is shown in Fig. 3.

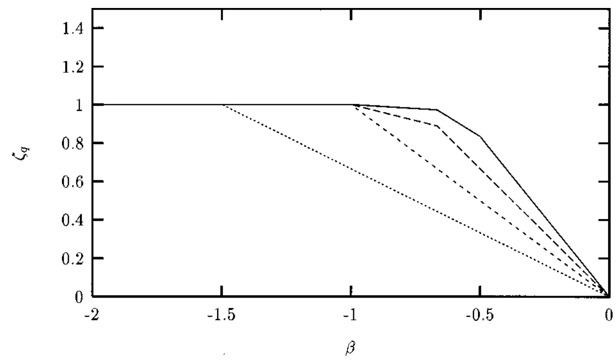

FIG. 3. Exponents ζ_2, ζ_3, ζ_4 and ζ_5 vs β, for $-2<\beta<0$. The lowest curve is that of ζ_2 and the ones above it are in the order ζ_3, ζ_4, and ζ_5. All the exponents stick at 1 for β sufficiently negative. As explained in the text, the exponents follow from our theoretical analysis of the equations for the structure functions.

Note that Eq. (12) implies that the dissipation scale depends on the order of the structure function being considered, a phenomenon characteristic of multifractality. Following conventional multifractal analysis [1] we can define an exponent h_4 by $\nu\sim\delta_4^{1+h_4}$ and from Eq. (12) we find $h_4=2(1+\beta)/3$ for $-1<\beta<-2/3$, which connects smoothly to the scaling exponent of $-\beta/3$ at $\beta=-2/3$. We have checked this behavior at $\beta=-0.8$ and $\beta=-0.4$ and the data for the former are shown in Fig. 4(a). The theoretical predic-

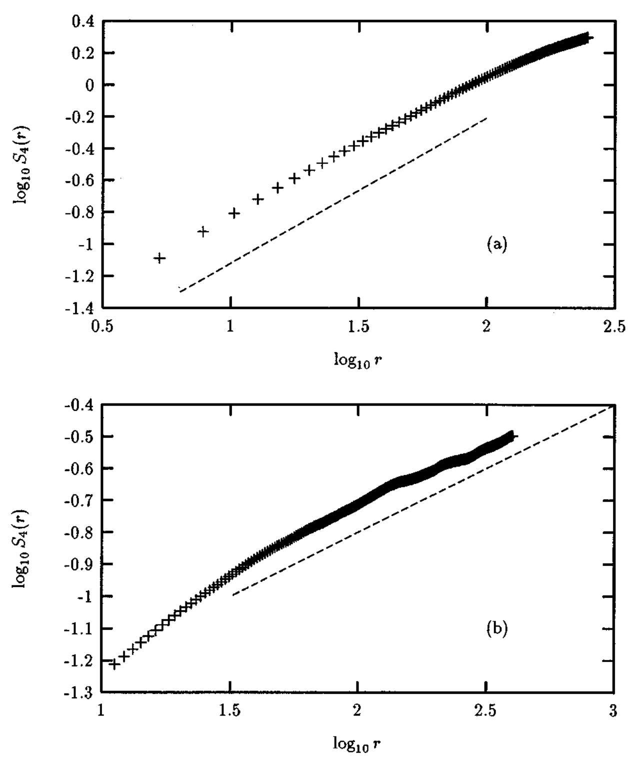

FIG. 4. (a) $\log_{10}S_4(r)$ vs $\log_{10}r$, for $\beta=-0.80$, $\nu=0.075$, and $D=1.0\times10^{-6}$. The straight line, drawn for comparison, has a slope equal to 0.91, while the predicted value is 0.9333. (b) $\log_{10}S_4(r)$ vs $\log_{10}r$, for $\beta=-0.30$, $\nu=0.06$, and $D=2.0\times10^{-6}$. The straight line, drawn for comparison, has a slope equal to $-4\beta/3$.

tion yields a value of 0.933 and our numerical results yield 0.90 ± 0.04. It becomes numerically difficult to extract the exponents reliably for $-0.3<\beta<0$ because the crossover length scales become large. The result for $\beta=-0.30$ is shown in Fig. 4(b). Our numerical simulations were done on a system of size $L=1024$ with $N=4096$ k modes in the spectral code. Some checks were made by halving the system size and doubling the number of modes. In addition, we have varied the value of ν by a factor of 3 and found that our results remain the same in the inertial range. Given that we do not use hyperviscosity it is difficult to reduce ν much further without a great deal more computer time. It is important to emphasize that our numerical results are accompanied by different consistency checks and are in quantitative accord with our theoretical analysis.

Our analysis of $S_5(r)$ is more involved although similar in spirit. We again assume that the behavior of $\langle[(\epsilon_1+\epsilon_2][u_1-u_2]^2\rangle$ in the $\nu\to0$ limit is dominated by the four point function $\langle\hat{u}(k_1)\hat{u}(k_2)\hat{u}(k_3)\hat{u}(k_4)\rangle$. The behavior deduced above for S_4 is used to conjecture forms for the four-point function allowing one to perform the analysis. For example, the symmetric version of the analog of Eq. (11) with the corresponding $\mu_1=\mu_2=\mu_3=\mu_4=-1+(1+\beta)/12$ that reproduces $S_4(r)\sim|r|^{(2-\beta)/3}$ is used in the regime $-1<\beta<-2/3$. This yields $\zeta_5=1-(1+\beta)/12$. There are now three different regions, one extending from $\beta=0$ to $\beta=-1/2$ where scaling behavior is obtained and another for $-2/3\leqslant\beta\leqslant-1/2$. The behavior of ζ_5 is shown in Fig. 3. The multifractal scaling exponent h_5 assumes the values $h_5=h=-\beta/3$ for $0>\beta>-1/2$, $h_5=1/2+2\beta/3$ for $-1/2>\beta>-2/3$, and $(1+\beta)/6$ for $-2/3>\beta>-1$. Note that both ζ_5 and h_5 are piecewise linear and continuous.

We conjecture the following scenario on the basis of the calculations outlined above. Simple scaling with $\zeta_q=-q\beta/3$ and $h=-\beta/3$ occurs over a progressively diminishing range of values of β as higher values of q are considered: the range is $-1<\beta<0$ for $q=3$, $-2/3<\beta<0$ for $q=4$, and $-1/2<\beta<0$ for $q=5$. Conversely, as $\beta\to0$, $\zeta_q=-q\beta/3$ for higher and higher values of q. For smaller values of β, there is multiscaling, which is different for different segments of β values. When one considers higher-order structure factors, we expect an increase in the number of segments of multiscaling, and the range of β where simple scaling holds decreases, its extremal point moving ever closer to 0. While the specific details depend on our analysis we can sum up our result as follows: at any β between 0 and -1, there is multiscaling, with a finite number of low-order structure factors obeying simple scaling, with that number increasing as $\beta\to0$.

We comment next on the standard scale-dependent dimension $D(h)$. Assuming that the exponents h_n we have determined from the $\nu\to0$ behavior are the ones that minimize the usual expression relating ζ_n to $D(h_n)$, namely [1],

$$\zeta_q=\min_q[qh_q+1-D(h_q)]. \tag{13}$$

One finds that $D=1$ for all q when simple scaling is obtained; this is consistent with the absence of shocklike structure at $\beta=0$. As β moves from 0 to -1, the fractal dimension relevant to each structure factor decreases in a piecewise continuous way from $D=1$ to $D=0$, a sign of the increasing role of shocks. For instance, as can be deduced from the values of ζ_4 and h_4 one has $D(4)=3(1+\beta)$ for $-1<\beta<-2/3$, and similarly, $D(5)=1+\beta$ for $-1<\beta<-2/3$. When $\beta=-1$, both $D(4)$ and $D(5)$ are equal to zero. For $\beta<-3/2$ the behavior is dominated by shocks; since for pure shocks one has $h=0$, corresponding to pointlike behavior and $\nu\propto\delta$ we find $\zeta_q=1$ for $q\geqslant2$.

Our investigations show that the Burgers equation is a very interesting laboratory for the study of certain theoretical issues related to turbulence.

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